Math 1A: Calculus **Instructor: Alexander Paulin**

Quiz 8: L'Hôpital's and Curve Sketching

Your name: *Discussions 201, 203 // 2018-11-02*

Throughout this **entire** quiz, we will be examining the function

$$f(x) = \frac{x^2 + e^x}{e^x}.$$

Please box $\boxed{\text{your answers}}$.

Problem 1 (1 point)**.** State the domain of f. All real numbers

Problem 2 (1 point)**.** Find the x and y intercepts of $y = f(x)$ (if they exist). $(0, 1)$ is y-intercept. There is no x-intercept because $x^2 + e^x > 0$ always.

Problem 3 (1 point)**.** This function is **even / odd /** $\boxed{\textbf{neither}}$ (circle one). It is **periodic /** $\boxed{\textbf{not periodic}}$ (circle one).

Problem 4 (4 points)**.** Find all asymptotes of $y = f(x)$, if there are any. If you compute a limit, make sure to show your work.

Solution: There cannot be any vertical asymptotes because the domain of f is $\mathbb{R}$. Let a denote either ∞ or $-\infty$:

$$\begin{aligned}
\lim_{x\to a} f(x) &= 1 + \lim_{x\to a} \frac{x^2}{e^x} && \text{indeterminate form} \\
&= 1 + \lim_{x\to a} \frac{2x}{e^x} && \text{indeterminate form} \\
&= 1 + \lim_{x\to a} \frac{2}{e^x} \\
&= \begin{cases} 1 & \text{if } a = \infty, \\ \infty & \text{if } a = -\infty. \end{cases}
\end{aligned}$$

So there is a $\boxed{\text{horizontal asymptote } y = 1 \text{ as } x \text{ tends to } \infty}$. There are no other asymptotes. □

Elaboration—not needed on quiz. If $y = mx + b$ were an asymptote, we should see either

$$\lim_{x\to\infty} (f(x) - (mx + b)) = 0 \quad \text{or} \quad \lim_{x\to-\infty} (f(x) - (mx + b)) = 0$$

I'll leave it to you to check (with L'Hôpital's Rule) that the only possibility is

$$\lim_{x\to\infty} (f(x) - (0 \cdot x + 1)) = 0$$

which is the horizontal asymptote we found above. In other words—there are no slant asymptotes. □

(Continued on back of sheet.)

Problem 5 (4 points)**.** Find all critical points (x, y) on the graph of $y = f(x)$. Using the first or second derivative test, determine whether they are local minima, local maxima, or neither.

Solution: Differentiating gives

$$\begin{aligned} f'(x) &= \frac{(2x + e^x)(e^x) - (x^2 + e^x)(e^x)}{(e^x)^2} && \text{quotient rule; remember that } \frac{d}{dx}e^x = e^x \\ &= \frac{2x - x^2}{e^x} && \text{simplify} \\ &= \frac{x(2-x)}{e^x}. \end{aligned}$$

The derivative always exists, so to find critical points we need only consider when $f'(c) = 0$. This happens when $c = 0, 2$.

If you chose to use the first derivative test: Examining the expression $\frac{x(2-x)}{e^x}$, we see that the denominator is always positive, while the sign of the numerator depends on x. To be specific:

$$f(x) \text{ is } \begin{cases} \text{negative} & \text{if } x < 0, \\ \text{positive} & \text{if } 0 < x < 2, \\ \text{negative} & \text{if } 2 < x. \end{cases}$$

Hence:

$(0, 1)$ is a local minimum, $(2, \frac{4}{e^2} + 1)$ is a local maximum, f is increasing on $(0, 2)$, and f is decreasing on $(-\infty, 0)$ and $(2, \infty)$.

If you chose to use the second derivative test: Differentiating again gives

$$\begin{aligned} f''(x) &= \frac{(2 - 2x)e^x - (2x - x^2)(e^x)}{(e^x)^2} && \text{quotient rule} \\ &= \frac{x^2 - 4x + 2}{e^x} && \text{simplify} \end{aligned}$$

and we see that $f''(0) = 2/e^2 > 0$ while $f''(2) = -2/e^2 < 0$. We conclude again that $(0, 1)$ is a local minimum and $(2, \frac{4}{e^2} + 1)$ is a local maximum. Then you can reason out the intervals on which f is increasing/decreasing by using the fact that f is continuous everywhere and that these are the only critical points. □

(a) State the intervals on which f is *increasing*: see above

(b) State the intervals on which f is *decreasing*: see above

(Don't worry about open vs. closed brackets for the intervals here.)

Problem 6 (4 points)**.** Report the x-coordinate of each inflection point of f. (You do not need to compute the y-coordinate.)

Solution: I computed f'' in the solution above. It always exists; we just need to examine where $f''(c) = 0$. The quadratic formula tells us that this happens when

$$c = \frac{4 \pm \sqrt{16 - 8}}{2} = 2 \pm \sqrt{2}.$$

Note that the parabola $x^2 - 4x + 2$ opens upwards. So:

$$f''(x) \text{ is } \begin{cases} \text{positive} & \text{if } x < 2 - \sqrt{2}, \\ \text{negative} & \text{if } 2 - \sqrt{2} < x < 2 + \sqrt{2}, \\ \text{positive} & \text{if } 2 + \sqrt{2} < x. \end{cases}$$

Because f changes concavity at $\boxed{2 - \sqrt{2} \text{ and } 2 + \sqrt{2}}$, these are (the x-coordinates of) the inflection points. □

(a) State the open intervals on which f is *concave up*: $(-\infty, 2 - \sqrt{2})$ and $(2 + \sqrt{2}, \infty)$

(b) State the open intervals on which f is *concave down*: $(2 - \sqrt{2}, 2 + \sqrt{2})$